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# Optimal Feedback Control: Foundations, Examples, and Experimental Results for a New Approach

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**Typical optimal feedback controls are nonsmooth functions. Nonsmooth controls raise fundamental theoretical problems on the existence and uniqueness of state trajectories. Many of these problems are frequently addressed in control applications through the concept of a Filippov solution. In recent years, the simpler concept of a $\pi$ solution has emerged as a practical and powerful means to address these theoretical issues. In this paper, we advance the notion of Carathéodory–$\pi$ solutions that stem from the equivalence between closed-loop and feedback trajectories. In recognizing that feedback controls are not necessarily closed-form expressions, we develop a sampling theorem that indicates that the Lipschitz constant of the dynamics is a fundamental sampling frequency. These ideas lead to a new set of foundations for achieving feedback wherein optimality principles are interwoven to achieve stability and system performance, whereas the computation of optimal controls is at the level of first principles. We demonstrate these principles by way of pseudospectral methods because these techniques can generate Carathéodory–$\pi$ solutions at a sufficiently fast sampling rate even when implemented in a MATLAB® environment running on legacy computer hardware. To facilitate an exposition of the proposed ideas to a wide audience, we introduce the core principles only and relegate the intricate details to numerous recent references. These principles are then applied to generate pseudospectral feedback controls for the slew maneuvering of NPSAT1, a spacecraft conceived, designed, and built at the Naval Postgraduate School and scheduled to be launched in fall 2007.**

## I. Introduction

CONSIDER the problem of generating feedback controls for the general nonlinear system

$$\dot{x} = f(x, u, t), \qquad u(t) \in \mathbb{U}(t, x(t)), \qquad x(t) \in \mathbb{X}(t) \tag{1}$$

where $t \twoheadrightarrow \mathbb{X}(t) \subset \mathbb{R}^{N_x}$ and $(t, x) \twoheadrightarrow \mathbb{U}(t, x) \subset \mathbb{R}^{N_u}$ are set-valued maps [1–3] for which the ranges are compact sets denoting the state and control spaces, respectively, and $f\colon \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \times \mathbb{R} \to \mathbb{R}^{N_x}$ is a control-parameterized vector field that satisfies $C^1$-Carathéodory conditions (see the Appendix) for every admissible control function $t \mapsto u$. We defer to Sec. II for a more detailed discussion of the meaning of a solution vis-à-vis the function spaces $\mathcal{X}$ and $\mathcal{U}$ that correspond to the state $x(\cdot)$ and control $u(\cdot)$ trajectories, respectively. For the moment, we shall treat $x(\cdot) \in \mathcal{X}$ and $u(\cdot) \in \mathcal{U}$ as part of the design specifications. The control space $\mathbb{U}(t, x)$ is allowed to be state-dependent, because these are important considerations in practical flight control [4,5] as well as in theory [1]. By feedback control, we mean a map $k\colon \mathbb{R} \times \mathbb{R}^{N_x} \to \mathbb{R}^{N_u}$, such that

$$k(t, x(t)) = u(t) \in \mathbb{U}(t, x(t)) \tag{2}$$

It is well recognized [1,2] that this is one of the most difficult problems in control theory. In this paper, we add two additional requirements to this problem. In the first specification, we require that the open-loop controls $t \mapsto u$ meet some specified optimality criterion of a Bolza-type cost functional:

$$J[x(\cdot), u(\cdot), t_0, t_f] := E(x(t_0), x(t_f), t_0, t_f) + \int_{t_0}^{t_f} F(x(t), u(t), t)\,dt \tag{3}$$

where $J\colon \mathcal{X} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the second specification requires that the initial and final states meet some specified endpoint conditions:

$$(x(t_0), x(t_f), t_0, t_f) \in \mathbb{E} \subset \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \times \mathbb{R} \times \mathbb{R} \tag{4}$$

where $t_0$ and $t_f$ are initial and final times under consideration. The time interval $t_f$–$t_0$ may be finite or infinite. Infinite-horizon problems are directly addressed in [6,7]. Thus, we are addressing a general problem of optimal feedback control for constrained nonlinear systems. Note that the cost functional is not necessarily limited to a quadratic criterion and the nonlinear dynamics are not limited to control-affine nonlinear systems. The limitations we impose on the system constraints will be clarified later, but it essentially reduces to articulating all the constraints in terms of functional inequalities in which the governing functions are smooth. This does not necessarily imply that the problem data (i.e., the stipulated sets in the problem definition) are smooth or that the functions describing these sets are smooth. A number of practical nonsmooth problems may be transformed (without simplification) to smooth problems by introducing additional control variables and constraints [8].

Our main motivation for considering such difficult problems is aerospace applications. Aerospace problems remain one of the most

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challenging problems in control theory, because optimality, in addition to the usual problems of constraints and nonlinearities, dictates the engineering feasibility of a space mission. For example, launch vehicle ascent guidance [9,10] and orbit control [8] problems demand fuel minimality as part of their engineering feasibility requirements. Even in problems in which such an obvious criterion is not stated, optimality is inherently required for feasibility, safety, and other considerations. For example, if a standard quaternion-based feedback control law is implemented for slewing the NPSAT1 spacecraft discussed in Sec. VII, it leads to an erroneous conclusion that a horizon-to-horizon scan is not feasible over an area of coverage, because the attitude maneuver time (about 50 min) exceeds the coverage time (about 10 min). On the other hand, a minimum-time maneuver is equal to about half the coverage time (5 min), indicating that a horizon-to-horizon scan is indeed physically realizable [11,12]. In other words, traditional feedback controls that are not based on optimality considerations may limit the performance of the system well under its true capabilities, as is evidently possible from the physics of the problem. Traditional feedback control laws may also diminish safety margins [13]. For example, one aspect of reentry safety is the size of the footprint: the larger the footprint, the safer the entry guidance, because it implies the availability of additional landing sites for exigency operations. Thus, footprint maximization is part of the entry guidance requirements [14]. Consequently, entry guidance algorithms that are not based on optimal control compromise safety [15]. It is apparent that both guidance and control problems in aerospace applications have optimality specified either directly or indirectly. Typically, vehicle guidance problems are finite-horizon problems, whereas stabilization problems are infinite-horizon control problems. In extending this concept for general nonlinear control systems (i.e., not necessarily aerospace), we follow Clarke [16] and note that there are essentially two principal objectives in all of control theory:

1) In the positional objective, the state trajectory $t \mapsto x$ must remain in or approach a given set of $\mathbb{R}^{N_x}$.

2) In the transfer objective, the control trajectory $t \mapsto u$ must transfer the state of the system from a given set in $\mathbb{R}^{N_x}$ to a target set in $\mathbb{R}^{N_x}$.

Traditionally, optimal control theory has been largely used to address the transfer objective. The positional (i.e., stability) objective can also be addressed by way of optimal control theory by linking the Bellman value function to a Lyapunov function, as facilitated by nonsmooth calculus [2,16]. For the type of feedback control promulgated in this paper, we have provided such a link in a complementary paper [6], but under a stronger assumption of smoothness of the Lyapunov function. In this paper, we outline the issues, principles, and certain implementation details. This allows us to discuss, at the very fundamental level, the key differences between traditional feedback control theory and our proposal outlined in Sec. IV. We support Sec. IV by developing a key Lemma in Sec. V that links the meaning of sampling frequency to the Lipschitz constant of the dynamic system. In Sec. VI, we derive a few simple rules of thumb to facilitate an implementation of our ideas. A demonstration of our ideas is discussed in Sec. VII by way of ground-test results for the time-optimal feedback control of NPSAT1 (see Fig. 1), an experimental spacecraft conceived, designed, and built at the Naval Postgraduate School and scheduled to be launched in fall 2007.

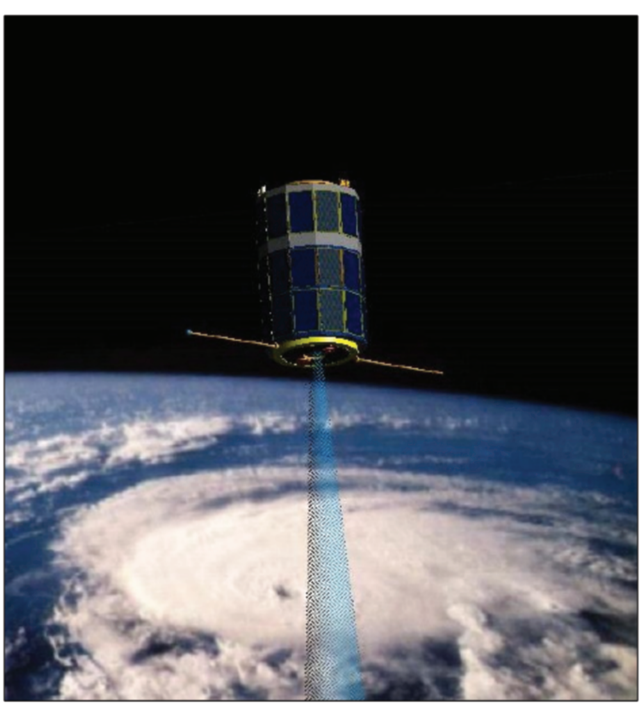

**Fig. 1 Artist's rendition of NPSAT1 in orbit.**

## II. Theoretical and Practical Issues in Optimal Feedback Control

A framework for solving the optimal feedback control problem posed in Sec. I is the well-known dynamic programming method. In this approach, we need to find a function $\varphi: \mathbb{R} \times \mathbb{R}^{N_x} \to \mathbb{R}$ that solves the Hamilton–Jacobi–Bellman (HJB) partial differential equation:

$$\mathcal{H}(\partial_x \varphi(t,x), x, t) + \partial_t \varphi(t,x) = 0 \quad \forall\ t \in [t_0, t_f] \tag{5}$$

where $\mathcal{H}$ is the lower Hamiltonian [2] given by

$$\mathcal{H}(\lambda, x, t) := \min_{u \in \mathbb{U}} H(\lambda, x, u, t) \tag{6}$$

$H$ is the control Hamiltonian,

$$H(\lambda, x, u, t) := F(x, u, t) + \lambda^T f(x, u, t) \tag{7}$$

and $\lambda \in \mathbb{R}^{N_x}$ is the covector $\partial_x \varphi(t,x)$. A point worth emphasizing at this juncture and not widely discussed in textbooks is the implication of Eq. (6) on Eq. (5). That is, to merely construct the HJB equation, we need to obtain a closed-form solution to the optimization problem

$$(\text{HMC}) \quad \begin{cases} \underset{u}{\text{Minimize}} & H(\lambda, x, u, t) \\ \text{Subject to} & u \in \mathbb{U}(t,x) \end{cases} \tag{8}$$

When $\mathbb{U}$ is prescribed in terms of functional inequalities, problem HMC (Hamiltonian minimization condition) is a nonlinear programming problem. Thus, to merely construct the HJB equation, the dynamic programming method requires a closed-form solution to a nonlinear programming problem! This is one reason why complex engineering problems seeking the HJB route are posed in a simplified manner that facilitates a closed-form solution to problem HMC. This is part of a prevailing philosophy in control theory of seeking exact solutions to simple problems. In striking contrast, our goals are to seek approximate, but highly accurate, solutions to complex, more realistic, problems. In any case, even under such simplifications, the HJB equation does not have a $C^1$-smooth solution for $\varphi$, even for simple problems. One of the earliest and best-known criticisms of the HJB approach is by Pontryagin et al. [17]. This issue was resolved in 1983 by Crandall and Lions [18], who developed the notion of viscosity solutions to the HJB equation. Recent updates to these ideas are described in [19]. Considering that obtaining even smooth solutions to partial differential equations is itself a very challenging problem, it is no surprise that obtaining viscosity solutions to the HJB equation are even more problematic. Finally, even if $\varphi$ is smooth, there is, in general, no continuous function $u = k(x)$ that satisfies the HJB equation:

$$F(x, k(x), t) + \partial_x \varphi^T f(x, k(x), t) + \partial_t \varphi(t,x) = 0 \tag{9}$$

Thus, even if all the preceding hurdles are overcome, we cannot hope to find a continuous optimal feedback law. That continuous feedback controls do not exist even for stabilizing control systems was proved by Brockett [20] as far back as 1983 in his famous counterexample of the nonholonomic integrator. Thus, in developing a general feedback control theory for the problem defined in Sec. I, we need to allow for the possibility of a discontinuous feedback law. When $k(x)$ is discontinuous, $f(x, k(x), t)$ is discontinuous, even if we had assumed $f(x, u, t)$ to be continuous in all its variables. Obviously, feedback is expected to alter the system dynamics, but a discontinuous feedback law changes the dynamics to such an extent that even the very definition of a solution $x(\cdot)$ for a discontinuous differential equation is called into question. This is because when $f(x, k(x), t)$ is continuous, we say that $x(\cdot)$ is a solution to the differential equation

$$\dot{x} = f(x, k(x), t) \tag{10}$$

if $\dot{x}(t)$ coincides with $f(x(t), k(x(t)), t)$ for all $t$. In the discontinuous case, Carathéodory's extension of this concept uses the "almost everywhere" moniker, together with $x(\cdot)$ being absolutely continuous (see the Appendix). This solution concept has served very well for open-loop systems,

$$\dot{x} = f(x, u(t), t) \tag{11}$$

and forms the basis of open-loop optimal control theory [17,21]; however, for closed-loop systems, it is well known that the Carathéodory-solution concept turns out to be quite unsatisfactory from the point of view of existence and uniqueness [22]. This is because Eq. (11) has discontinuities, at most, in the independent variable $t$, whereas Eq. (10) allows discontinuities in the dependent variable $x$ as well. Hence, the question of providing a suitable definition of a solution for discontinuous differential equations is of fundamental importance. This theoretical problem clearly affects practice: for example, in the difficulties encountered in seeking to numerically simulate a trajectory. Among the large variety of definitions proposed in the literature, the Filippov-solution concept [22] is widely used in control applications. The Filippov solution is based on a convexification of the differential inclusion associated with a differential equation. In 1994, Ryan [23] and Coron and Rosier [24] extended Brockett's [20] conclusion by proving the validity of his result for discontinuous feedback if $x(\cdot)$ is a Filippov solution. As a result of such outstanding issues in discontinuous differential equations, vigorous research in this field continues (see, for example, [25–27] and the references contained therein).

In recent years, a neoclassic notion of a solution to a discontinuous differential equation has been advanced as a means to address such long-standing theoretical difficulties. This solution concept is based on closed-loop system sampling. That is, suppose we want to find the solution to a nonsmooth differential equation:

$$\dot{x} = g(x), \qquad x(0) = x_0$$

This is recast in the form of a feedback control

$$u = g(x)$$

to the controlled differential equation

$$\dot{x} = u, \qquad x(0) = x_0$$

and the ideas of Sec. III are applied. Not only is this idea different from the Filippov- or Carathéodory-solution concepts, it is also one of the most obvious and practical ways of engineering a digital solution to a controlled, and hence ordinary, differential equation. In other words, what has happened in recent years is the close juxtaposition of practical methods for control with advance mathematics, particularly nonsmooth calculus. This is in sharp distinction to the developments of control theory of prior years that were largely focused on seeking closed-form or analytical expressions for feedback representations $k(x)$, wherein $k$ was designed by means of elementary operations (e.g., multiply and add) over elementary programmable functions (e.g., polynomials and transcendental representations). In the concept proposed in this paper, $k(x)$ is designed and computed by way of designer functions that have no discernible standard representations. Thus, theory and computation are now fundamentally intertwined in first principles. A preliminary stability theory from this perspective is provided in our companion paper [6]; here, we outline the basic principles.

## III. $\pi$ Trajectory: Definition and Perspectives

Although the $\pi$ trajectory originated as a result of digital computer technology, it has become clear in recent years that the concept is quite useful as a fundamental mathematical object in addressing the issues outlined in Sec. II. To illustrate this point, we first consider time-invariant feedback control laws $\tilde{k}\colon \mathbb{R}^{N_x} \to \mathbb{R}^{N_u}$ applied to autonomous dynamic systems $\dot{x} = f(x, u)$. In the next section, we consider the more general *time-varying* feedback law $k\colon \mathbb{R} \times \mathbb{R}^{N_x} \to \mathbb{R}^{N_u}$, because this is central to our new approach.

In following Clarke [16], we let $\pi = \{t_i\}_{i\geq 0}$ be a partition of the time interval $[0, \infty)$, with $t_i \to \infty$ as $i \to \infty$. The quantity

$$\operatorname{diam}(\pi) := \sup_{i\geq 0}(t_{i+1} - t_i)$$

is called the diameter of $\pi$. Given an initial condition $x_0$, a $\pi$ trajectory $x(\cdot)$, corresponding to $\tilde{k}(x)$, is defined as follows [16]: From $t_0$ to $t_1$, we generate a classical solution to the differential equation:

$$\dot{x}(t) = f(x(t), \tilde{k}(x_0)), \qquad x(t_0) = x_0, \qquad t_0 \leq t \leq t_1 \tag{12}$$

A classical solution exists for Eq. (12) because $f$ is Lipschitz-continuous with respect to its first argument. At $t = t_1$, we set $x_1 = x(t_1)$ and restart the system with $u = \tilde{k}(x_1)$:

$$\dot{x}(t) = f(x(t), \tilde{k}(x_1)), \qquad x(t_1) = x_1, \qquad t_1 \leq t \leq t_2$$

Continuing in this sample-and-hold manner, we end up with a closed-loop trajectory corresponding to a piecewise-constant open-loop control. *Note that this $\pi$-solution concept is not to be confused with an Euler polygonal arc*. Under the concept of a $\pi$ trajectory, it is possible to rigorously prove a rich number of theorems that have practical consequences. These theorems directly connect the epsilons and deltas of abstract mathematics to practical digital computer implementations of optimal control [16,28]. An implementation of the sample-and-hold feedback concept is illustrated in Fig. 2. To maintain the focus of this paper on the main elements of feedback control, we assume state information to be available on demand. The problem of nonlinear state estimation is addressed elsewhere [29,30]. The switch in Fig. 2 remains open over open time intervals, $(t_{i+1}–t_i)$, and is closed only over zero measure (in time). It is clear that a $\pi$ trajectory is indeed equivalent to a zero-order-hold (ZOH) digital implementation of a continuous-time input signal; however, note that *in the $\pi$-solution concept, the $\pi$ trajectories are viewed as fundamental objects that are not explicitly reliant upon a digital computer*. To drive home this point, we note that we would consider $\pi$ solutions even if we had analog computers only (i.e., even if digital computers were nonexistent). In this case, a $\pi$ solution would be implemented by the switch mechanism illustrated in Fig. 2. Thus, in a mathematical analysis of solutions to differential equations, we now consider sequences of $\pi$ trajectories parameterized by $\operatorname{diam}(\pi)$. In principle, the limiting situation $\operatorname{diam}(\pi) \to 0$ can then be connected to traditional feedback control theory; however, when traditional control theory is implemented via a digital computer, it is clear that the practice of the two concepts

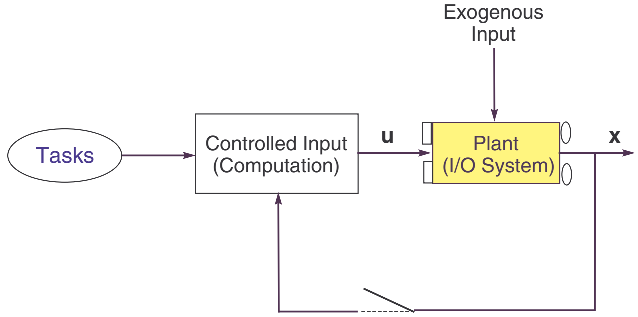


**Fig. 2 Illustrating closed-loop control by a sampling technique.**

merge before $\text{diam}(\pi) \to 0$. Thus, in the new perspective promoted here, we discard the limiting condition $\text{diam}(\pi) \to 0$ in favor of an analysis to consider what happens when $\text{diam}(\pi) < \delta_\pi$, where $\delta_\pi > 0$ is a given number. *Under this philosophy, we can no longer demand exactness*. Thus, for example, rather than demand $\|x(t)\| \to 0$ as $t \to \infty$ as in standard asymptotic stability theory, we only require that there exist some $T > 0$ such that $\|x(t)\| < \epsilon_x \ \forall\, t \geq T$, where $\epsilon_x > 0$ is a specified tolerance. This is the notion of practical stability [31,32] introduced in the 1960s. In adopting this concept, we require that $\|x(t)\| < \epsilon_x \ \forall\, \epsilon_x \geq \epsilon_{\text{ses}} > 0$, where $\epsilon_{\text{ses}} > 0$ is a given number based on sensor and estimation errors. Even under simulation, $\epsilon_x \neq 0$ as a result of computational errors. As further elaborated in later sections of this paper, these new concepts illustrate how theory and computation have become intertwined at the fundamental level.

As a matter of comparison, we note that a ZOH implementation of a standard constant gain feedback control law $u = Kx$, where $K$ is some gain matrix, generates a $\pi$ trajectory. It is therefore clear that a *practical (digital) implementation of any closed-loop controller involves an open-loop strategy*. The most widely used open-loop strategy is a *hold* implementation. This concept is illustrated in Fig. 3. Also shown in Fig. 3 is a practical modification of the $\pi$ trajectory when the time to compute $\tilde{k}(x)$ is nonnegligible as can happen with legacy systems employing old computer technology. An extension of the ZOH concept is the first-order hold. In a first-order (or higher-order) hold, $u(t)$ is obtained by extrapolation to the future from the past. Such implementations require memory storage of an appropriate number of previous measurements to perform the extrapolation. *Thus, a second point to note here is that a natural extension of this concept generates feedback implementation policies that are a backward-looking approximation to the continuous-time signal* $u(t) = \tilde{k}(x(t))$. That is, we have a staircase approximation for a zero-order hold, a (backward) secant approximation in a first-order hold, etc.

In summary, a sample-and-hold feedback controller generates a $\pi$ trajectory as a fundamental theoretical and practical solution to a controlled differential equation whether or not the feedback controller is smooth. Note that these ideas do not address the determination of $\tilde{k}(x)$; thus, finding a feedback map that guarantees stability of the closed-loop system continues to dominate ongoing research in nonlinear control theory.

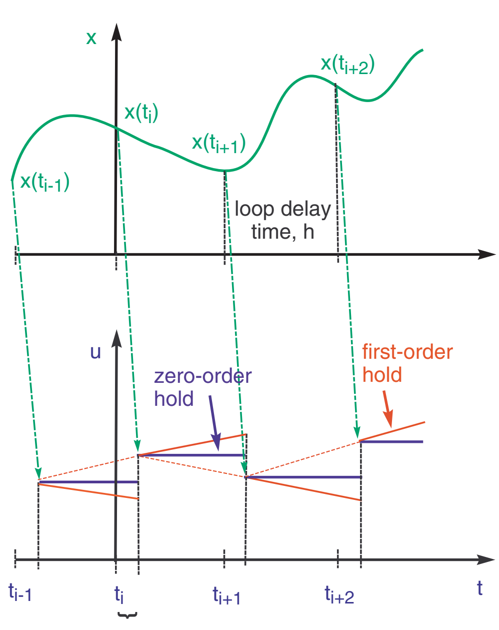


**Fig. 3 Standard sampling implementations of control signals illustrating the backward-looking feedback policy; also shown are computational times that are assumed to be less than the time delay in receiving sensor updates.**

## IV. Carathéodory–$\pi$ Trajectory: A Forward-Looking Approach to Feedback

Now suppose that the feedback law was time-varying: $u = k(t, x)$. In principle, we could implement it in exactly the same style as suggested in Sec. III and all the prior conclusions would hold. That is, we could continue to implement a sample-and-hold procedure,

$$u(t) = k(t_i, x(t_i)), \qquad t \in [t_i, t_{i+1}] \tag{13}$$

or its standard extensions to a first- or higher-order hold. We would indeed arrive at this conclusion by the standard procedure [17] in optimal control theory in dealing with nonautonomous systems by introducing a new state variable, $\hat{x} \in \mathbb{R}$, such that

$$\dot{\hat{x}} = 1, \qquad \hat{x}(t_0) = t_0 \tag{14}$$

By augmenting Eq. (14) with the dynamics, as in

$$\dot{x} = f(x, u, \hat{x}), \qquad \dot{\hat{x}} = 1$$

we can define a new state, $\tilde{x} = (x, \hat{x})$, and transform $k(t, x)$ quite simply to $\tilde{k}(\tilde{x}) = k(\hat{x}, x)$. Sussmann [33] noted that this procedure, adopted by Pontryagin et al. [17] to address nonautonomous systems, is quite limiting from a theoretical standpoint, because Eq. (14) assumes that $f(x, u, t)$ is of class $C^1$ with respect to $t$, a much stronger requirement than $f$, being merely measurable in $t$, a sufficient condition for the existence of a Carathéodory solution. Here, we propose that this idea is also quite unsatisfactory for practical feedback control. This is because a time-varying feedback law $k(t, x)$ inherently contains an element of prediction as a result of its explicit dependence on the clock time $t$. Hence, it is clear that a better policy for control between samples is a forward-looking implementation:

$$u(t) = k(t, x(t_i)), \qquad t \in [t_i, t_{i+1}] \tag{15}$$

This strategy does not pose the problem of the meaning of a solution for a discontinuous differential equation,

$$\dot{x} = f(x, k(t, x(t_i)), t)$$

because the discontinuities between the samples are now only with respect to the independent variable $t$ and not the dependent variable $x$. Thus, any possible discontinuities we have to contend with are in the open-loop segments over the sample time. Hence, we can define a solution over the sample segment in the standard Carathéodory sense and then glue all these Carathéodory-solution pieces in the same manner as the $\pi$ trajectory. Hence, we distinguish this concept as a Carathéodory–$\pi$ trajectory (see Fig. 4).

The reason for choosing Carathéodory segments and not, for instance, the Filippov solution is motivated by optimal control considerations. That is, we will later design $k(t, x)$ using optimal control principles and generate the open-loop segments by computing open-loop optimal controls. One motivation for optimality is, as indicated in Sec. I, aerospace problems. Regardless, optimal control also has the advantage of guaranteeing stability through the Bellman value function serving as a natural Lyapunov function [6]. Thus, the existence of an open-loop optimal control is now directly tied to constructing closed-loop solutions. As already noted in Sec. II, a rich number of theorems are available for optimal open-loop trajectories defined in the Carathéodory sense. Consequently, we are naturally led to a Carathéodory modification of the $\pi$ trajectory.

The practical consequence of a Carathéodory–$\pi$ trajectory is that over any time segment $[t_i, t_{i+1}]$, the control is continuously updated by the clock information $t$. This implementation implies that the controlled input box in Fig. 2 (i.e., the computation box) includes a memory storage device for dumping the signal $u(t) = k(t, x(t_i))$ $t \in [t_i, t_{i+1}]$ that is to be fed to the plant between samples (see Fig. 5). The potentially extra cost of this new implementation compared with the ZOH policy is a clock and a memory storage device to store the signal $u(t)$, $t \in [t_i, t_{i+1}]$. Note the key difference between this policy and a higher-order hold. In a higher-order hold implementation

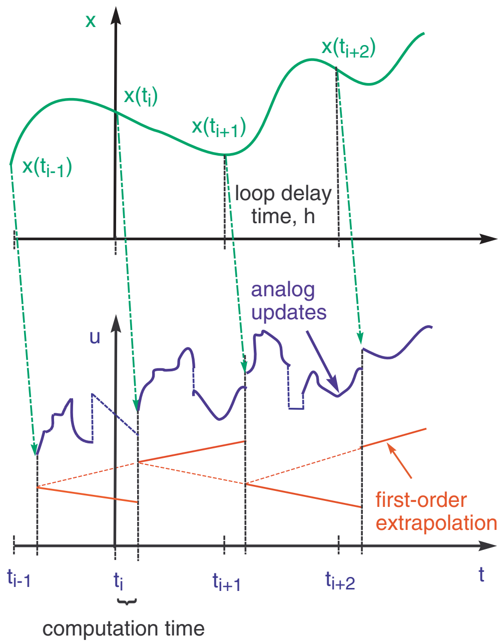


**Fig. 4 A proposed implementation of control signals in a clock-based forward-looking feedback policy for the generation of Carathéodory–$\pi$ solutions and its contrast to the backward-looking first-order hold.**

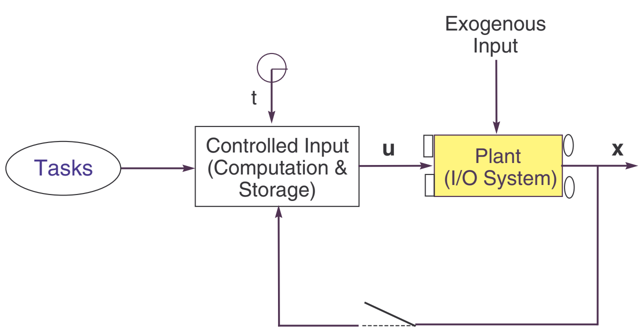


**Fig. 5 Implementing a clock-based feedback controller via computation and storage of open-loop controls.**

(which also requires memory storage), the control values between samples is based on the *past controls*, with current values obtained based on *extrapolation*. In the implementation of Eq. (15), the control signal is based on *current controls*, the currency being indicated by the clock time. These points are illustrated in Fig. 4. Note that it is not possible to use a clock-based feedback implementation if the feedback control law is independent of the clock time, as in $u = \tilde{k}(x)$. Thus, a time-varying feedback control law $u = k(t, x)$ is integral to our ideas.

At first glance, the clock-based feedback implementation may appear to deliver a minimal gain in performance at significant expense. The reason this is not necessarily true is that with regard to Eq. (13), $\delta_\pi$ must be substantially smaller than the diameter of $\pi$ required of Eq. (15) for any given $\epsilon_x$. This is a direct effect of using the clock information in the feedback law $k(t, x)$. Deferring the details of this discussion to Sec. V, the practical implications of Eq. (15) is that we have traded the requirement of a small $\delta_\pi$ to the purchase of a clock and a memory unit. The higher allowable $\delta_\pi$ now implies that we can afford to have a more sophisticated feedback strategy (e.g., optimal) than traditional analytical expressions for $k$. A computation of this sophisticated feedback law $u = k(t, x)$ is now permitted to have a sampling time far greater than that afforded by Eq. (13). This allows us to abandon the notion of seeking closed-form expressions for $k$ and resort to its fundamental form, $k\colon \mathbb{R} \times \mathbb{R}^{N_x} \to \mathbb{R}^{N_u}$, where the map is given by some computational algorithm. In essence, a large value of $\delta_\pi$ is now used to support the computation time for the control algorithm. Hence, we add an explicit clock to Fig. 2, as shown in Fig. 5, to emphasize the shift in philosophy that clock times are used to generate open-loop analog signals $u(t) = k(t, x(t_i))$, thereby subverting the traditional concept of real time as something equivalent to a very small $\delta_\pi$. Intuitively, it is clear that the requirement on the sampling frequency depends upon the time constant of the plant dynamics, among other things. The Lipschitz constant of $f$ acts as this fundamental time constant, as developed in the next section.

## V. Feedback-Based Requirements on Sampling Frequency

Consider a nonlinear *model* of a control system,

$$\dot{x} = f(x, u, t; p_0) \tag{16}$$

where $p_0 \in \mathbb{R}^{N_p}$ is a constant of system parameters (such as inertia, mass, etc.). The purpose of feedback is largely to manage various uncertainties such as imperfections in plant modeling, unmodeled/unknown exogenous input (see Fig. 5), estimation errors, etc. To this end, we let $t \mapsto \zeta(t)$ be some function in $L^\infty_{\text{loc}}$, so that

$$\dot{x} = f(x, u, t; p) + \zeta(t) \tag{17}$$

represents the dynamics of the real system (plant). In Eq. (17), $p$ denotes the actual plant parameters. Thus $p = p_0$ and $\|\zeta\|_{L^\infty} = 0$ corresponds to perfect modeling. We implement a feedback policy as follows (see Fig. 6). At the sampling time $t = t_i$, we determine $[t_i, t_f] \mapsto u = k(t, x_R(t_i))$, where $x_R$ is the state of the real system (plant). In the infinite-horizon case, we find $[t_i, \infty) \mapsto u$ by way of a domain transformation technique [6,7]. In any event, under the action of an open-loop control $[t_i, t_{i+1}] \mapsto k(t, x_R(t_i))$, the state of the model at $t_{i+1}$ is given by

$$x_M(t_{i+1}) = x_R(t_i) + \int_{t_i}^{t_{i+1}} f(x_M(t), k(t, x_R(t_i)), t; p_0)\,dt \tag{18}$$

Let

$$\tau_i := t_{i+1} - t_i \tag{19}$$

be the representative sampling time (not necessarily a constant), such that $t_{i+1}$ is the instant when the control $t \mapsto k(t, x_R(t_i))$ is available for application to the plant (see Fig. 6). Then the state of the plant at $t_{i+1}$ is determined by the action of the control $[t_i, t_{i+1}] \mapsto k(t, x_R(t_{i-1}))$ and is given by

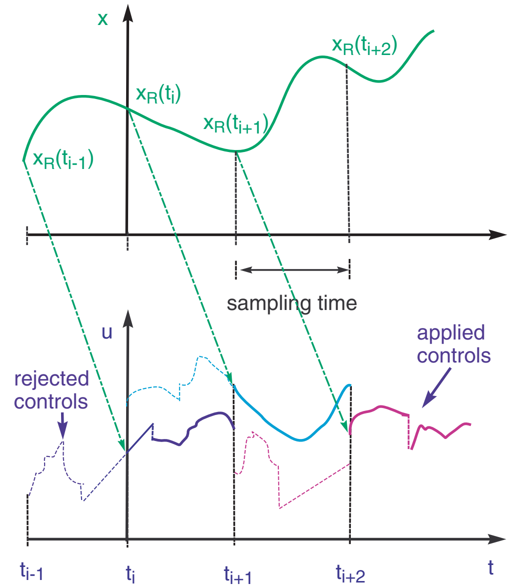


**Fig. 6 Practical implementation of control signals in clock-based feedback control; the control $u(t) = k(t, x_R(t_i))$, $t \in [t_i, t_{i+1}]$ is computed but not applied.**

$$x_R(t_{i+1}) = x_R(t_i) + \int_{t_i}^{t_{i+1}} f(x_R(t), k(t, x_R(t_{i-1})), t; p)dt + \int_{t_i}^{t_{i+1}} \zeta(t)dt \quad (20)$$

That is, $[t_i, t_{i+1}] \mapsto x_R$ differs from the ideal/model trajectory $[t_i, t_{i+1}] \mapsto x_M$, due to sampling effects as well as deviations caused by $t \mapsto \zeta(t)$ and $p$. Subtracting Eqs. (18) and (20), we have

$$x_R(t_{i+1}) - x_M(t_{i+1}) = \int_{t_i}^{t_{i+1}} f(x_R(t), k(t, x_R(t_{i-1})), t; p)dt - \int_{t_i}^{t_{i+1}} f(x_M(t), k(t, x_R(t_i)), t; p_0)dt + \int_{t_i}^{t_{i+1}} \zeta(t)dt \quad (21)$$

Although it is possible to estimate an upper bound for the right-hand side of Eq. (21) under existing assumptions, a stronger result is possible under the following additional assumption.

*Assumption 1*:

1) For each $x$ and almost all $t$, the function $(u, p) \mapsto f(x, u, t; p)$ is Lipschitz-continuous.

2) For each $x$, the function $t \mapsto k(t, x)$ is in $L^\infty_{\mathrm{loc}}$.

Under Assumption 1, from Eq. (21), we have

$$\|x_R(t_{i+1}) - x_M(t_{i+1})\| \le \mathrm{Lip} f_x \int_{t_i}^{t_{i+1}} \|x_R(t) - x_M(t)\|dt + \mathrm{Lip} f_u \int_{t_i}^{t_{i+1}} \|k(t, x_R(t_{i-1})) - k(t, x_R(t_i))\|dt + \mathrm{Lip} f_p \|p - p_0\|\tau_i + \|\zeta\|_{L^\infty}\tau_i \quad (22)$$

where $\mathrm{Lip} f$ denotes the Lipschitz constants of $f$ with respect to the subscripted variables.

*Lemma 1*: Let $\delta > 0$ be the smallest number, such that

1)

$$\|k(\cdot, x_R(t_{i-1})) - k(\cdot, x_R(t_i))\|_{L^\infty} \le \alpha_1 \delta \frac{\mathrm{Lip} f_x}{\mathrm{Lip} f_u}$$

2)

$$\|\zeta\|_{L^\infty} \le \alpha_2 \delta \, \mathrm{Lip} f_x$$

3)

$$\|p - p_0\| \le \alpha_3 \delta \frac{\mathrm{Lip} f_x}{\mathrm{Lip} f_p}$$

for some $\alpha_i \in (0, 1)$ and

$$\sum_i \alpha_i = 1$$

Then, for any given $\epsilon > 0$, if

$$\tau_i \le \frac{W(r)}{\mathrm{Lip} f_x} \quad (23)$$

it implies that $\|x_R(t_{i+1}) - x_M(t_{i+1})\| \le \epsilon$, where $W(r)$ is the Lambert $W$ function (see the Appendix), and $r$ is the ratio $r := \epsilon/\delta$.

*Proof*: From Eq. (22) and conditions 1, 2, and 3 of the Lemma, we have

$$\|x_R(t_{i+1}) - x_M(t_{i+1})\| \le \mathrm{Lip} f_x \int_{t_i}^{t_{i+1}} \|x_R(t) - x_M(t)\|dt + \delta \, \mathrm{Lip} f_x \tau_i \quad (24)$$

From Gronwall's Lemma (see the Appendix), Eq. (24) reduces to

$$\|x_R(t_{i+1}) - x_M(t_{i+1})\| \le \delta \, \mathrm{Lip} f_x \tau_i \exp(\mathrm{Lip} f_x \tau_i) \quad (25)$$

From the definition of the Lambert $W$ function (see the Appendix), it can be easily verified that for $y, z \in \mathbb{R}_+$, if $z \le W(y)$, then $ze^z \le y$; hence, from Eqs. (23) and (25), we have

$$\|x_R(t_{i+1}) - x_M(t_{i+1})\| \le r\delta = (\epsilon/\delta)\delta = \epsilon$$

*Remark 1*: Under perfection, that is, if $\|\zeta\|_{L^\infty} = 0$ and $\|p - p_0\| = 0$, the only contribution to $\delta$ is from condition 1 of the lemma. If $t \mapsto k(t, x(t))$ is an optimal control, then it follows from Bellman's principle of optimality and local uniqueness that $k(t, x(t_{i-1})) = k(t, x(t_i))$. Hence, under perfection and optimality, $\delta = 0$ for arbitrary sampling times. Hence, $r \to \infty$ for any $\epsilon > 0$. Because $W(r) \to \infty$ as $r \to \infty$, Eq. (23) conforms to the fact that under ideal conditions, open-loop optimal controls are sufficient if these controls are generated to meet performance and constraint specifications.

*Remark 2*: Under the conditions of Remark 1, we have $\delta \ne 0$ for sample-and-hold feedback, except under the special situation $k(t, x(t_{i-1})) = k(t, x(t_i)) = \text{constant} \; \forall \, t_i$. This is one reason why the Carathéodory–$\pi$ feedback solution permits a longer computational delay than the $\pi$ solution.

*Remark 3*: The clock-based feedback controller has an inherent safety component in the following sense: If the feedback signals (wires) were cut off in Fig. 5, open-loop constrained optimal control signals remain as plant inputs, in contrast to a constant signal of a ZOH implementation.

### A. Applications of Lemma 1

As noted in Sec. I, one of the objectives of control theory is positional. That is, for the positional objective of stabilizing a system about the origin, we require

$$\|x_R(t)\| \le \epsilon_{\mathrm{Req}} \quad \forall \; t \ge T \quad (26)$$

where $\epsilon_{\mathrm{Req}} > 0$ is a given number based on design requirements and is consistent with the accuracy of the state estimator (and hence sensor errors). It is well known [16] that stability can be achieved by optimal control principles. Hence, if optimal controls are generated at the rate specified by Eq. (23), stability can be achieved under appropriate technical conditions. A proof of such a theorem, along with several example problems, are discussed in [6]. Thus, Eq. (23) provides a sufficient condition for the definition of *real time* in real-time optimal control.

For finite-horizon feedback control (as in vehicle guidance), we require that $x_R(t_i)$ $t_i \in \pi$ be such that for each $x_R(t_i)$, there exist open-loop controls $[t_i, t_f] \mapsto k(t, x_R(t_i))$, such that at time $t_f$,

$$(x_R(t_f), t_f) \in \mathbb{E}_f \subset \mathbb{E}$$

Thus, if open-loop optimal controls are generated starting at $t = t_0$, it is possible to guide a vehicle to its target by generating a Carathéodory–$\pi$ feedback solution. An example of such a strategy for entry vehicle guidance is developed in [5].

### B. Connections to Continuous-Valued Feedback Control

It is worth noting that we never find or compute $(t, x) \mapsto k(t, x)$ $\forall \, x \in \mathbb{X}$; rather, we only compute its "semidiscrete, semianalog" representation in the sense that we find $k(t, x)$ at only discrete points in state space but allow $k(t, x)$ to be computed at all times $t$. Thus, even if there were to be a unique continuous-valued feedback controller, we only consider its semidiscrete representations.

From a theoretical point of view, we may consider limit points of convergent subsequences of these semidiscrete solutions as forming generalized solutions to nonsmooth differential equations $\dot{x} = f(x, k(t, x), t)$, $x(t_0) = x^0$. This is an open research problem. This point, once again, illustrates the convergence of theory with practice!

### C. Connections to Prior Ideas

The developments of the preceding sections were motivated by several fundamental concepts that are strung together to provide a fresh outlook on feedback. For instance, although Lemma 1 is a new result, it makes intuitive sense that the Lipschitz constant of the dynamics is a nonlinear generalization of the (linear) concept of a time constant. This development arose from our definition of a Carathéodory–$\pi$ solution, which was motivated by the definition of a $\pi$ solution, which, in turn, was motivated by a mathematical definition of a solution to nonsmooth differential equations, and so on. Clearly, each one of the many concepts proposed in the preceding sections have roots in previously proposed ideas. For the purpose of brevity, we do not delineate all these roots but illustrate a few connections to highlight some of the origins of the ideas proposed in this paper.

That a closed-form solution is a sufficient but not a necessary condition for feedback can be traced back to Pontryagin et al. , who describe the problem of synthesizing optimal controls using high-speed computers and relay circuits to generate closed-loop (i.e., feedback) solutions using open-loop optimal controls. Although they did not discuss sampling issues, their intuitive concept has been widely used within the aerospace community since the 1960s: for example, in powered explicit guidance [34], which employs open-loop linear tangent steering in a closed-loop manner. Based on similar ideas, several problems are solved in the classic text by Bryson and Ho [35]. Motivated by advancements in computer technology, these concepts migrated in the 1990s to stabilization problems in electrical and chemical engineering under the heading of model predictive control. Clearly, the Carathéodory–$\pi$ solution concept has its roots in all these prior ideas. What the concept offers is a formalization of the notion of a feedback solution in terms of closed-loop system sampling as a fundamental mathematical concept that is not driven by advancements in computer technology. Thus, if a sampling frequency meets the sufficient condition put forth by Lemma 1, the control generation may be called real-time and its implementation would meet the definition of a feedback controller. The resulting function $k(t, x)$ is a designer function, in that $k(t, x)$ is not formed by elementary operations over elementary functions as in a typical closed-form solution [36–38]; rather, to each control problem, $k(t, x)$ is a special input–output function for which the design and computation are the fundamental problems of feedback control. We design $k(t, x)$ based on optimal control principles, and its computation is largely transparent to the designer through the Carathéodory–$\pi$ concept, in much the same manner as the coordinate rotation digital computer (CORDIC) algorithm [39] is transparent to a user employing a pocket calculator to compute trigonometric functions. In other words, the concept of a designer function $k(t, x)$ is similar to how we perceive trigonometric functions as analytical functions, despite the fact that it is computed by the CORDIC algorithm on most digital computing machines.

## VI. Practical Implementation of the Principles

The principles developed in the previous section deal with the original control problem head on. That is, instead of seeking to minimize quadratic cost functions that may have no practical bearing on the actual control system or to simplify the dynamic model of the plant to derive closed-form solutions, we propose to deal directly with the original problem itself. As pointed out in Sec. I, our motivations for this route largely stem from aerospace applications. Philosophically, what we seek are approximate solutions to the exact (i.e., original) problem, rather than exact solutions to the approximate problem. This leads us to the following precepts:

### A. Precision in Practical Control

The accuracy afforded by the sensors/estimation provides a lower limit on the precision attainable by any control system, regardless of the sophistication of the control law. That is, no amount of sophistication in control theory can achieve precision finer than the resolution of the sensors. Thus, if the precision of the sensors/estimation is $\epsilon_{\text{ses}} > 0$ in $x$ (e.g., $3\sigma$ value), then it is impossible to (verifiably) achieve any stability requirement that has $\epsilon_{\text{Req}} < \epsilon_{\text{ses}}$. Alternatively, for a given precision requirement, we must choose sensors with precision, such that $\epsilon_{\text{ses}} \leq \epsilon_{\text{Req}}$. Typically, we choose $\epsilon_{\text{ses}} \approx 0.1\epsilon_{\text{Req}}$; that is, we choose sensors to be about an order of magnitude finer in sensing than the requirement imposed on stability. Thus, this implies that if we seek to transfer a system from an initial point $x_0 = x^0$ to a final point $x_f = x^f$, we replace this notion, without prejudice, by seeking controls that transfer the state from an initial epsilon set, such as an epsilon ball,

$$x_0 \in \mathbb{B}(x^0, \epsilon_{\text{ses}}) \tag{27}$$

to a final epsilon set,

$$x_f \in \mathbb{B}(x^f, \epsilon_{\text{Req}}) \tag{28}$$

where $\mathbb{B}(x^*, r)$ denotes a closed ball of radius $r$ centered around $x^*$:

$$\mathbb{B}\,(x^*, r) := \{x\colon \|x - x^*\| \leq r\} \tag{29}$$

This point once again illustrates the merging of approximation theory with control theory at the fundamental level. In Sec. VII, we revisit such epsilon sets explicitly, with the epsilon sets defined in terms of the infinity norm.

### B. Impact of Computational Inexactness on Precision

In demonstrating the preceding principles for general nonlinear systems by way of numerical computations, we note that it is quite impossible to obtain exact solutions. That is, numerical calculations automatically introduce $\zeta(t)\not\equiv 0$ in Eq. (17); hence, a perfect numerical simulation of a perfect plant is quite impossible. Alternatively, we may regard numerical errors as simulating state estimation errors with $\epsilon_{\text{ses}} = \epsilon_{\text{num}} > 0$, associated with a perfect plant (i.e., model) and imperfect measurements. We use this notion explicitly in Sec. VII. Thus, traditional theoretical issues such as $\epsilon \to 0$ must be replaced by other epsilon–delta concepts; for instance, the existence of a $\delta$ such that for all $\epsilon \geq \epsilon^* > 0$, something happens (see Lemma 1). Furthermore, because the central point of feedback is to manage uncertainties, the very notion of an exact solution is an anathema, because if exact modeling were possible, open-loop control would suffice and feedback would be unnecessary (also a consequence of Lemma 1). Consequently, the fact that our methods rely on approximations is not to be construed as a shortcoming; rather, we contend that approximations are essential and inherent for the design of feedback control itself. What we have done, and demonstrate experimentally in Sec. VII, is to bring computational principles to the very foundations of control theory, rather than as an afterthought. Thus, quantities such as $\epsilon_{\text{num}}, \epsilon_{\text{ses}}, \epsilon_{\text{Req}}$, etc., must be properly coordinated with the physics of the problem in a manner that a mathematically realizable solution exists if a physically realizable solution is possible. This is essentially a problem of designing the control system.

### C. Computation of Real-Time Open-Loop Controls

An implementation of the principles described in Secs. III, IV, and V requires a sufficiently fast generation of open-loop controls. Thus, if open-loop controls can be generated as demanded by Eq. (23), closed loop is achieved quite simply by generating Carathéodory–$\pi$ solutions. In recent years, it has become quite apparent that pseudospectral (PS) methods [40–48] are capable of generating optimal open-loop controls within fractions of a second [41,48,49], even when implemented in legacy hardware running MATLAB. This implies that real-time optimal controls can be generated for systems with large Lipschitz constants: that is, systems with fast dynamics. This simply follows by rewriting Eq. (23) as

$$\operatorname{Lip} f_x \leq \frac{W(r)}{\tau_c}$$

where $\tau_c$ is the largest computational time. Nonlinear systems for which such PS-based feedback controls have been demonstrated range from the classical problem of swinging up a pendulum [6] to satellite attitude control [11] to reentry vehicle guidance [5]. *Solutions to more than a hundred problems are documented in [43] and the references contained therein*. In Sec. VII, we provide experimental results that demonstrate this concept for the attitude control of NPSAT1.

PS methods for computing optimal controls are readily available by means of the software package DIDO [50]. DIDO is a minimalist's approach to solving optimal control problems. Only the problem formulation described in Sec. I is required. DIDO incorporates the covector mapping principle [40,51–54], which essentially allows dualization to commute with discretization so that the necessary conditions for optimality can be automatically verified.

### D. Generation of Time-Smooth Controls

In general, the optimal controls that generate the Carathéodory–$\pi$ solutions are nonsmooth with respect to time (see Figs. 4 and 6). This nonsmoothness arises due to two key facts:

1) The controls are allowed to be discontinuous between sampling times.

2) The controls are allowed to be discontinuous at the sampling times.

The discontinuity at the sampling time is no different from those that occur in the standard ZOH controller; however, the discontinuity between the sampling time is allowed to achieve high performance (i.e., optimality), as motivated by problems in aerospace and introduced in Sec. I. In certain applications, discontinuous controls may not be desirable and smooth control may be stated as a requirement. As noted earlier, the traditional approach to designing such control systems is by way of choosing quadratic cost functions. This work-around fails in our proposed method, because it can only prevent discontinuities between sampling times and not the discontinuities at the sampling time. One option is to impose continuity of the controls at the sampling times. Imposing such continuity conditions has two drawbacks:

1) The controls are, at best, continuous and may continue to be nonsmooth with respect to time.

2) A theoretical analysis of the system would be unnecessarily difficult, due to the differences between the posed problem and the implemented solution.

A far simpler approach to generating time-smooth optimal controls is by a direct imposition of smoothness, as facilitated by the notion of Sobolev spaces (see the Appendix): the home space for optimal control theory. Also see [8] for a simple and practical illustration of the utility of Sobolev spaces.

In open-loop optimal control theory, $\mathcal{U} = L^\infty$, and nonsmoothness in $t \mapsto u$ can arise as a result of optimality. Suppose we require that $u(\cdot)$ be smoother, say, $u(\cdot) \in W^{1,\infty}$; then, we simply extend the dynamic system such that the new state space is a subset of $\mathbb{R}^{N_x} \times \mathbb{R}^{N_u}$, with the new states given by

$$\bar{x} = [x^T, u^T]$$

The dynamics of the original states are, as before, given by $\dot{x} = f(x, u, t)$. The dynamics for the "state" $u$ are now given by

$$\dot{u} = \bar{u} \tag{30}$$

where $\bar{u}$ is the new control variable that is chosen from some appropriate compact set:

$$\bar{u} \in \bar{\mathbb{U}} \subset \mathbb{R}^{N_u}$$

Because we automatically require that $\bar{u}(\cdot) \in L^\infty$, Eq. (30) guarantees that $u(\cdot) \in W^{1,\infty}$. In other words, Eq. (30) ensures in a single stroke that the control trajectory for the original controls, $t \mapsto u$, will be smooth at both the sampling time and between samples. The inertia introduced for the original controls ($u$, by way of $\bar{\mathbb{U}}$) is at the discretion of the designer. Clearly, higher-order smoothness may be attained in a similar fashion. Because the new problem has a new vector field, $[f(x, u), \bar{u}]$, a theoretical analysis can be carried out in concert with practical implementation; that is, there would be no gap between theory and practice under this implementation. *Note, however, that if the original problem had only pure control constraints, the new problem would have both state and control constraints*. This is one additional reason why state-constrained optimal control problems are important even if the original state space $\mathbb{X}(t)$ was an open set in $\mathbb{R}^{N_x}$ for each $t$. Such implementations of control generation have been performed successfully for a number of problems (see [5,55,56]).

## VII. Example: Non-Eulerian Spacecraft Slew Control

In this section, we demonstrate a successful experimental application of the theory developed in the previous sections for slewing NPSAT1 (see Fig. 1). An air-bearing setup of the spacecraft for ground testing is shown in Fig. 7. NPSAT1 uses magnetic actuators and a pitch momentum wheel for attitude control. One experiment onboard the NPSAT1 spacecraft is to demonstrate, in flight, the application of the new principles developed in the previous sections. One component of this experiment is the performance of the minimum-time feedback controller when the pitch momentum wheel is caged or failed. Thus, the cost function is simply given by the transfer time and the attitude motion is governed by magnetic torquers.

### A. Dynamic Model for the Air-Bearing Setup

Choosing the standard quaternion and body rates as the state variables, we have $x = (q, \omega) \in \mathbb{R}^7$, where $q = (q_1, q_2, q_3, q_4)$ is the quaternion of the body frame with respect to the laboratory frame and $\omega = (\omega_x, \omega_y, \omega_z)$ is the angular velocity of the body frame with respect to the laboratory frame expressed in the body frame.

We suppress many of the differential geometric concepts involved in the motion of a spacecraft, because all of these notions have a differential algebraic counterpart. Thus, for example, when we write $q \in \mathbb{R}^4$, we do not imply that $q$ takes values in all of $\mathbb{R}^4$ but that $q$ is simply a vector of four variables.

It is quite straightforward [30] to show that the dynamic equations of motion for the NPSAT1 air-bearing setup are given by

$$\dot{q}_1(t) = \tfrac{1}{2}[\omega_x(t)q_4(t) - \omega_y(t)q_3(t) + \omega_z(t)q_2(t)] \tag{31}$$

$$\dot{q}_2(t) = \tfrac{1}{2}[\omega_x(t)q_3(t) + \omega_y(t)q_4(t) - \omega_z(t)q_1(t)] \tag{32}$$

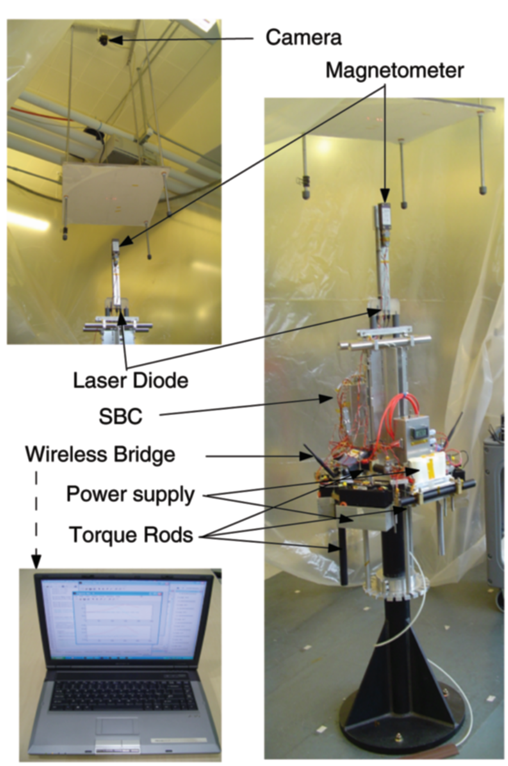


**Fig. 7 Air-bearing setup for the ground test of NPSAT1.**

$$\dot{q}_3(t) = \tfrac{1}{2}[-\omega_x(t)q_2(t) + \omega_y(t)q_1(t) + \omega_z(t)q_4(t)] \quad (33)$$

$$\dot{q}_4(t) = \tfrac{1}{2}[-\omega_x(t)q_1(t) - \omega_y(t)q_2(t) - \omega_z(t)q_3(t)] \quad (34)$$

$$\dot{\omega}_x(t) = \frac{I_2 - I_3}{I_1}\omega_y(t)\omega_z(t) - \frac{mgl}{I_1}C_{23}(q(t)) + \frac{1}{I_1}[B_z(q(t),t)u_2(t) - B_y(q(t),t)u_3(t)] \quad (35)$$

$$\dot{\omega}_y(t) = \frac{I_3 - I_1}{I_2}\omega_x(t)\omega_z(t) + \frac{mgl}{I_2}C_{13}(q(t)) + \frac{1}{I_2}[B_x(q(t),t)u_3(t) - B_z(q(t),t)u_1(t)] \quad (36)$$

$$\dot{\omega}_z(t) = \frac{I_1 - I_2}{I_3}\omega_x(t)\omega_y(t) + \frac{1}{I_3}[B_y(q(t),t)u_1(t) - B_x(q(t),t)u_2(t)] \quad (37)$$

where $(I_1, I_2, I_3) = (2.60, 2.87, 1.45)\ \text{kg}\cdot\text{m}^2$ are the principal moments of inertia of the NPSAT1 air-bearing assembly; $m = 59.0$ kg is the mass of the table platform; $g$ is the gravitational constant; $l = 0.56$ mm is the distance from the center of mass of the table to the center of rotation; $C_{ij}(q)$ is the quaternion-parameterized $ij$th element of the direction cosine matrix

$$C(q) = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2, 2(q_1q_2 + q_3q_4), 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4), q_2^2 - q_1^2 - q_3^2 + q_4^2, 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4), 2(q_2q_3 - q_1q_4), q_3^2 - q_1^2 - q_2^2 + q_4^2 \end{bmatrix}$$

$(B_x(q,t), B_y(q,t), B_z(q,t))$ are the components of the Earth's magnetic field in the body frame,

$$\begin{pmatrix} B_x(q,t) \\ B_y(q,t) \\ B_z(q,t) \end{pmatrix} = C(q)\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \end{pmatrix} \quad (38)$$

and $(B_1(t), B_2(t), B_3(t))$ are the components of the Earth's magnetic field in the laboratory frame. A detailed description of the measurements of this field at the Naval Postgraduate School's NPSAT1 Laboratory are described in [57]. Based on these measurements, the magnetic field was averaged to the constant values

$$(B_1(t), B_2(t), B_3(t)) = (0.24970, 0.09993, 0.39280)\ \text{G}$$

That is, given the various uncertainties imbedded in the NPSAT1 ground-test assembly, it was determined that no substantial advantages would be obtained by taking laboratory fluctuations in the magnetic field. For on-orbit applications, however, the $B$ field is taken to be time-varying and determined from the dipole model:

$$B_1(t) = \frac{M_e}{r_0^3}[\cos(\omega_0 t)[\cos(\epsilon)\sin(i) - \sin(\epsilon)\cos(i)\cos(\omega_e t)] - \sin(\omega_0 t)\sin(\epsilon)\sin(\omega_e t)]$$

$$B_2(t) = -\frac{M_e}{r_0^3}[\cos(\epsilon)\cos(i) + \sin(\epsilon)\sin(i)\cos(\omega_e t)]$$

$$B_3(t) = \frac{2M_e}{r_0^3}[\sin(\omega_0 t)[\cos(\epsilon)\sin(i) - \sin(\epsilon)\cos(i)\cos(\omega_e t)] + 2\cos(\omega_0 t)\sin(\epsilon)\sin(\omega_e t)]$$

where $M_e = 7.943 \times 10^{15}$ Wb · m is the magnetic dipole moment of the Earth, $\epsilon = 11.7$ deg is the magnetic dipole tilt, $i = 35.4$ deg is the orbit inclination of NPSAT1, and $\omega_e = 7.29 \times 10^{-5}$ rad/s is the spin rate of the Earth. Note also that there are other differences between the air-bearing and flight models of NPSAT1; for instance, the $mgl$ terms in Eqs. (35–37) are replaced by gravity gradient torques [11,30].

The controls $(u_1, u_2, u_3) \in \mathbb{R}^3$ are the actuator dipole moments on NPSAT1. Clearly, the dynamics of NPSAT1 are quite complex, with substantial nonlinearities.

### B. State and Control Spaces

That the quaternion variables must lie on $S^3$ generates a state space given by

$$\mathbb{X} = \{(q,\omega) \in \mathbb{R}^7\colon \|q\|_2 = 1, q_4 \geq 0\}$$

The controls on NPSAT1 are bounded by the maximum available dipole moment; this generates a control space given by

$$\mathbb{U} = \{u \in \mathbb{R}^3\colon \|u\|_\infty \leq 33\ \text{A}\cdot\text{m}^2\}$$

Thus, the NPSAT1 control system contains both state and control constraints.

In accordance with Eq. (1), we have provided a mathematical model for the dynamic system, obviously a nonlinear system. Furthermore, because the dipole moment is directly controlled by passing current to the solenoid, it is essentially inertialess; hence, $\mathcal{U} = L^\infty([t_0, t_f], \mathbb{R}^3)$. Thus, to each $u(\cdot) \in L^\infty$, $f$ satisfies $C^1$-Carathéodory conditions.

### C. Slew Control Problem

As briefly noted in Sec. I, the transfer objective of NPSAT1 is to slew the spacecraft in minimum time. Thus, the cost function for slew is simply given by the transfer time:

$$J_T[x(\cdot), u(\cdot), t_T] := t_T - t_0$$

The optimal control problem is to maneuver NPSAT1 in minimum time from a rest-to-rest state. A benchmark set of infinite-precision boundary conditions are given by

$$[q_1(t_0), q_2(t_0), q_3(t_0), q_4(t_0)] = [0, 0, \sin(\phi_0/2), \cos(\phi_0/2)] \equiv [q_1^0, q_2^0, q_3^0, q_4^0]$$
$$[\omega_x(t_0), \omega_y(t_0), \omega_z(t_0)] = [0,0,0] \equiv [\omega_x^0, \omega_y^0, \omega_z^0]$$
$$[q_1(t_T), q_2(t_T), q_3(t_T), q_4(t_T)] = [0, 0, \sin(\phi_f/2), \cos(\phi_f/2)] \equiv [q_1^f, q_2^f, q_3^f, q_4^f]$$
$$[\omega_x(t_T), \omega_y(t_T), \omega_z(t_T)] = [0,0,0] \equiv [\omega_x^f, \omega_y^f, \omega_z^f]$$

with $\phi_0 = 0$ and $\phi_f = 135$ deg.

As noted in Sec. VII, the boundary conditions can only be attained within the context of sensor/estimation precision. Based on NPSAT1's sensor and estimation accuracies, the practical boundary conditions on NPSAT1 for a 135-deg slew are given by

$$q_i(t_0) \in \mathbb{B}(q_i^0, 0.1\varepsilon_{q_i}), \qquad q_i(t_f) \in \mathbb{B}(q_i^f, \varepsilon_{q_i}), \qquad i = 1,2,3,4 \quad (39)$$

$$\omega_j(t_0) \in \mathbb{B}(\omega_i^0, 0.1\varepsilon_\omega), \qquad \omega_j(t_f) \in \mathbb{B}(\omega_f^0, \varepsilon_\omega), \qquad j = x,y,z \quad (40)$$

where

$$(\varepsilon_{q_1}, \varepsilon_{q_2}, \varepsilon_{q_3}, \varepsilon_{q_4}) = (0.087, 0.087, 0.034, 0.081)$$

and $\varepsilon_\omega = 0.0025$/s. These numbers are based on the capability of sensors onboard NPSAT1 in detecting deviations from the target set of point conditions to about a tenth of the mission requirements. This is why the initial conditions are known to within the sensor capability, whereas the final conditions are targeted to the specified requirements. Complete details on the sensors and filtering algorithms are discussed in [30].

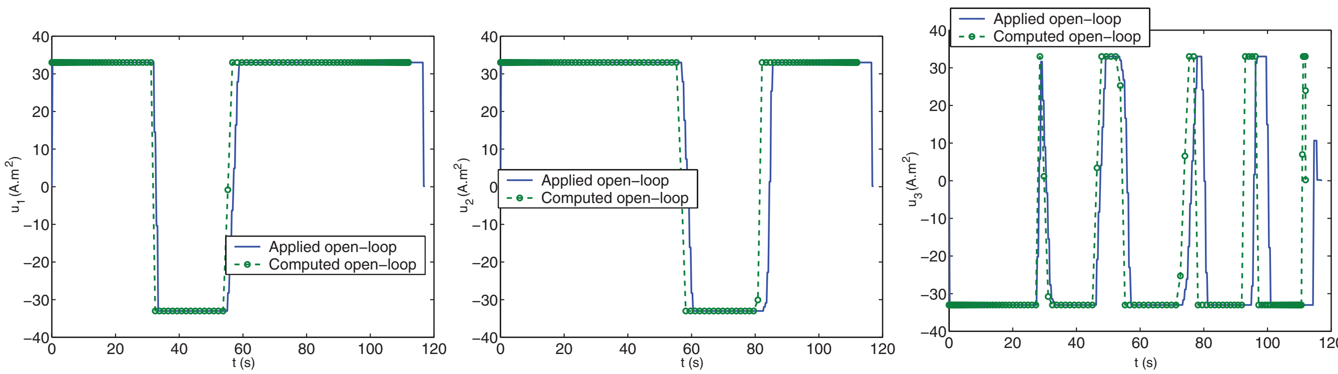


**Fig. 8 PS-computed open-loop optimal controls $t \mapsto u_i$ ($i = 1, 2, 3$) for the NPSAT1 air-bearing model.**

### D. Experimental Results

Shown in Fig. 8 are the time-optimal open-loop controls generated for the NPSAT1 ground-test model. These controls were obtained using PS methods by way of DIDO [50]; that is, the computation box indicated in Fig. 5 was DIDO. The controls are optimal in the sense that they satisfy all the necessary conditions for optimality, as demonstrated in [11,12]; hence, strictly speaking, they are only extremals. For the purposes of brevity, we do not describe these optimality tests because they are extensively discussed elsewhere [11,12]. Also shown in Fig. 8 are the open-loop controls applied to the experimental setup. They are slightly shifted from the computed controls because of various time-delay issues such as the wireless communication (see Fig. 7) to the onboard single-board computer (SBC) on NPSAT1. The quaternion responses of the spacecraft, both model-expected and experimentally achieved, are shown in Fig. 9. As expected, the open-loop controls drive the model to the target conditions, but the actual experiment fails to meet the requirement, as a result of uncertainties and imperfections, the hallmark of any engineering setup. In the case of NPSAT1, the uncertainties can be attributed to $p$ and $\zeta(t)$ in Lemma 1. That is, the values of the plant parameters $I_1$, $I_2$, etc., are not known exactly, and in the development of the dynamic model of the plant in Sec. VII.A, we assumed that the magnetic torquers were aligned with the principal axes. The last assumption is tantamount to a $\zeta(t)$-type disturbance.

The closed-loop control trajectories generated along the lines detailed in Sec. IV are shown in Fig. 10. Also shown in Fig. 10 are the open-loop controls for the purpose of comparison. Having interacted with the plant (i.e., experimental setup), the closed-loop control trajectories are indeed substantially different from the open-loop controls. The closed-loop quaternion responses are shown in Fig. 11. Closed-loop was achieved by a full-state feedback based on the state estimates obtained by an unscented Kalman filter (UKF). Implementation details of the UKF are fully described in [30]. It is apparent from Fig. 11 that the closed-loop response does not track the

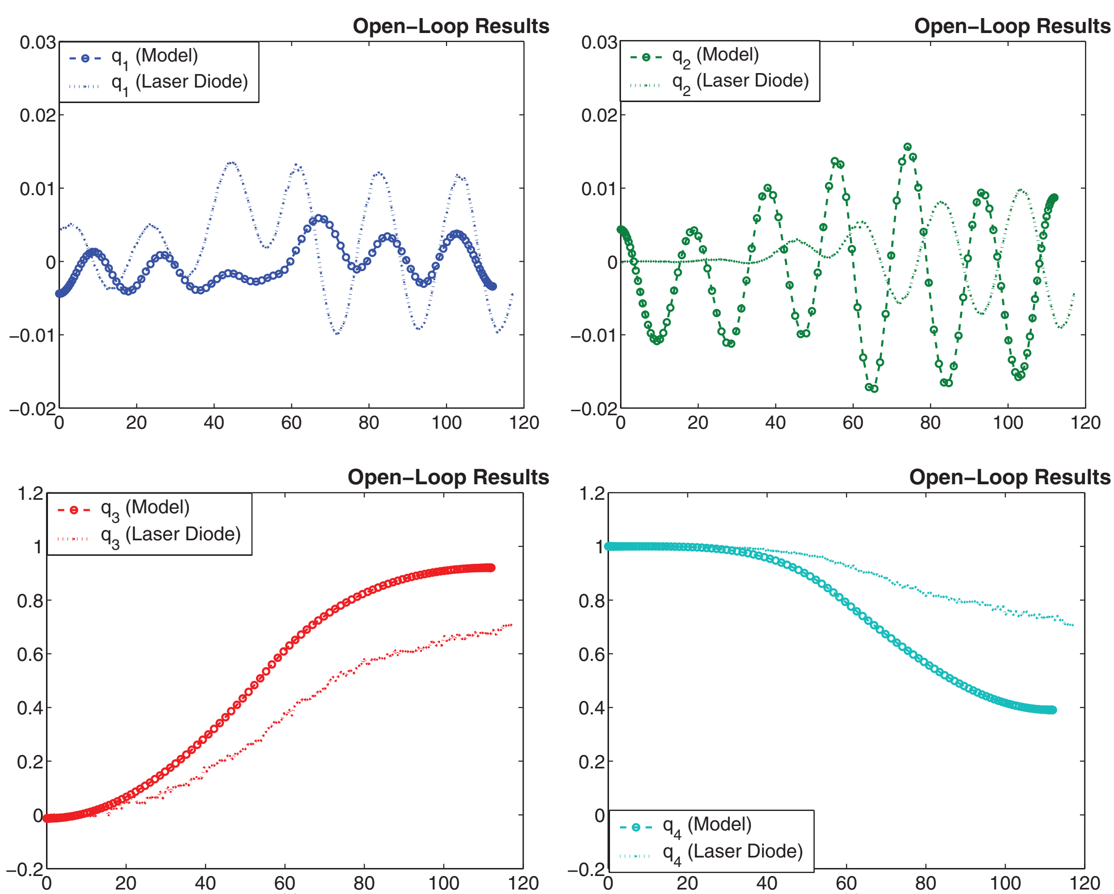


**Fig. 9 PS-computed open-loop minimum-time quaternion trajectory $t \mapsto q_i$ ($i = 1, 2, 3, 4$) for the NPSAT1 air-bearing model and experimental setup shown in Fig. 7.**

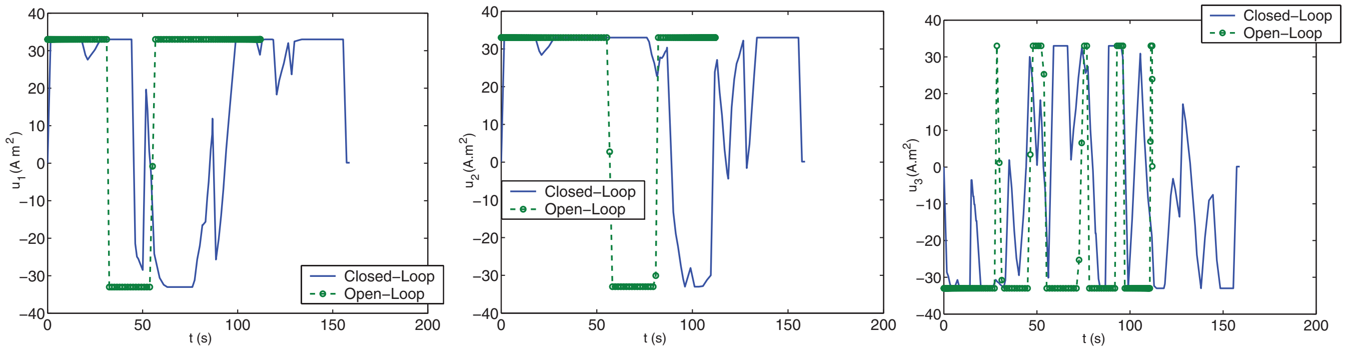


**Fig. 10 PS-computed closed-loop control trajectories $t \mapsto u_i$ ($i = 1, 2, 3$) for the NPSAT1 air-bearing experimental setup; compare Fig. 6.**

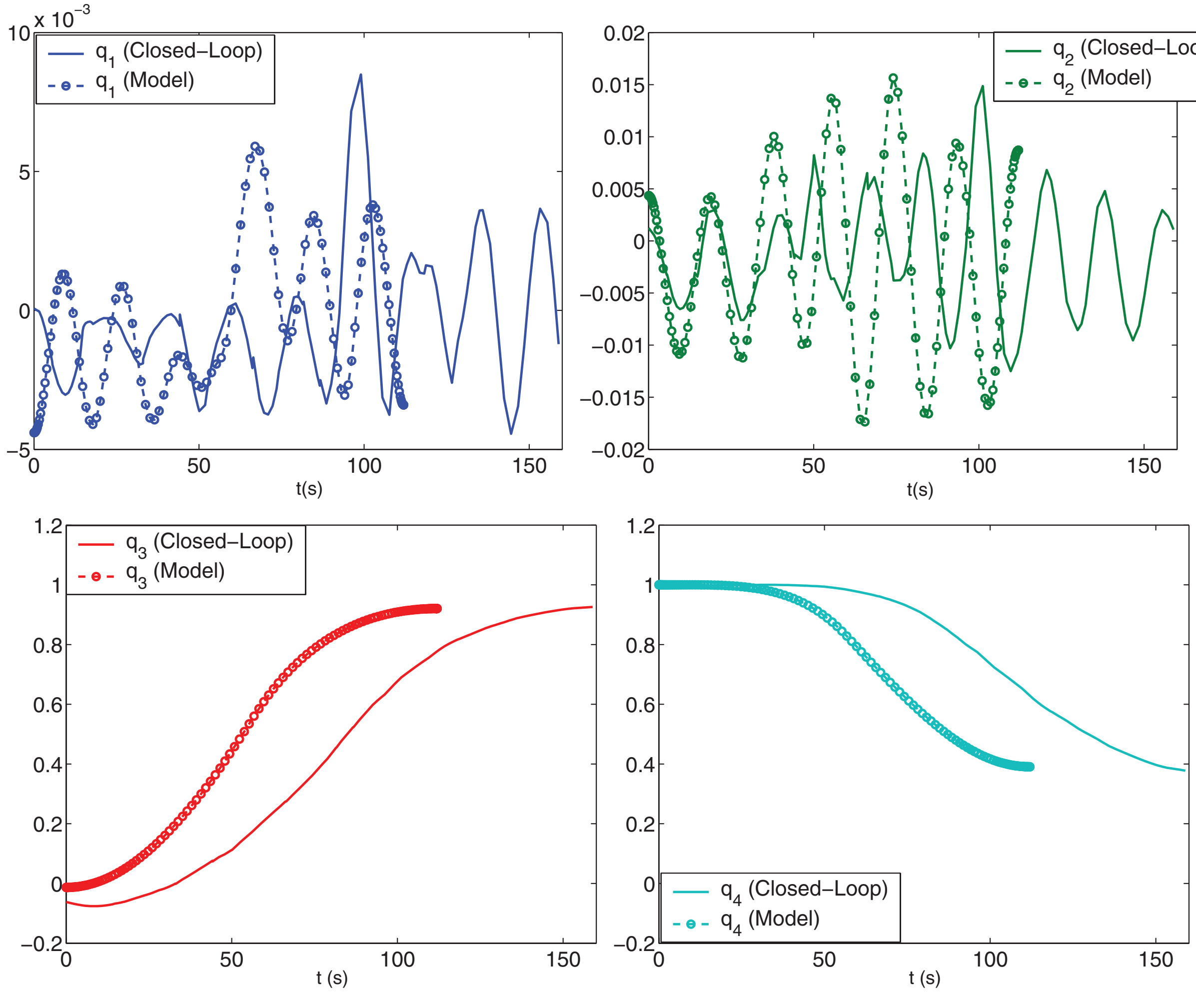


**Fig. 11 Closed-loop Carathéodory–$\pi$ trajectories $t \mapsto q_i$ ($i = 1, 2, 3, 4$) for the NPSAT1 air-bearing experimental setup illustrating the substantial differences between the plant and model trajectories.**

optimal trajectory of the model; in fact, tracking the time-optimal reference trajectory would be impossible, because the closed-loop plant would not be able to slew within 112 s (the minimum-time solution for the model). This is because the minimum-time solution (in the sense of Carathéodory–$\pi$) for the plant is 159 s. The obvious reasons for the differences in the optimal solution between the plant and the model are parameter uncertainties and modeling errors. Feedback is expected to manage these uncertainties and errors, and Fig. 11 illustrates that the closed-loop response does indeed just that in achieving the target conditions stipulated by Eqs. (39) and (40). These points illustrate one of the many ways in which our control system design is quite different from those advocated in standard control theory; that is, we simply deal with the original specifications directly. Should the original specifications call for a tracking controller, this could be achieved, but through a different problem definition (see, for example, [58], which also contains flight results for a PS-tracking controller implemented onboard the International Space Station).

As a matter of completeness, the angular velocity response of the closed-loop system is shown in Fig. 12. It is clear that the final rest conditions are indeed obtained well within the requirements of the $\omega$ epsilon ball. Finally, as a point of further validation of the experimental results, the online sensing and estimation of the local magnetic field is shown in Fig. 13. In addition to validating the excellent performance of the UKF, this plot illustrates how magnetic sensing and actuation could be performed at the same time.

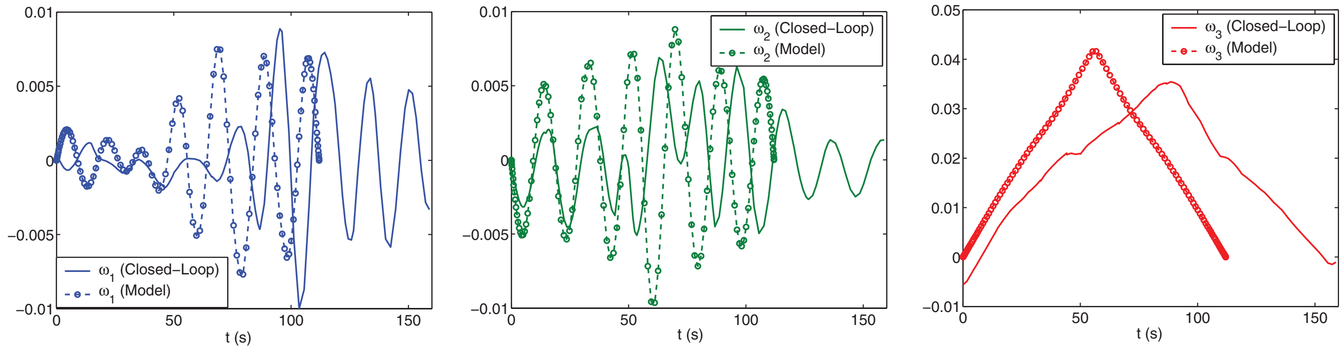


**Fig. 12 Closed-loop Carathéodory–$\pi$ trajectories $t \mapsto w_i$ ($i = 1, 2, 3$) for the NPSAT1 air-bearing experimental setup illustrating the substantial difference between the plant and model trajectories.**

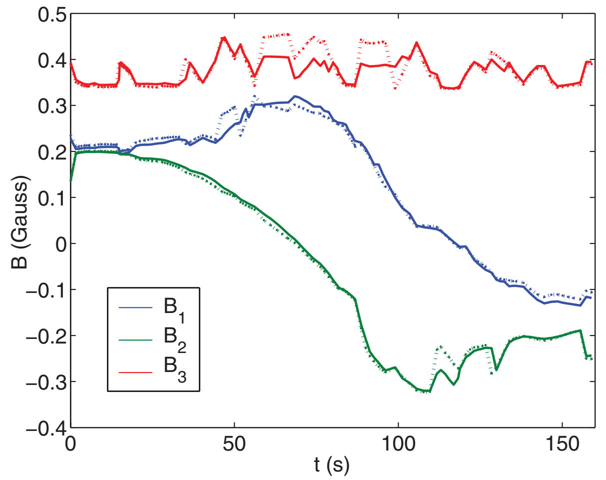


**Fig. 13 Sensed (dashed line) and UKF-estimated (solid line) local magnetic field during closed-loop control of NPSAT1.**

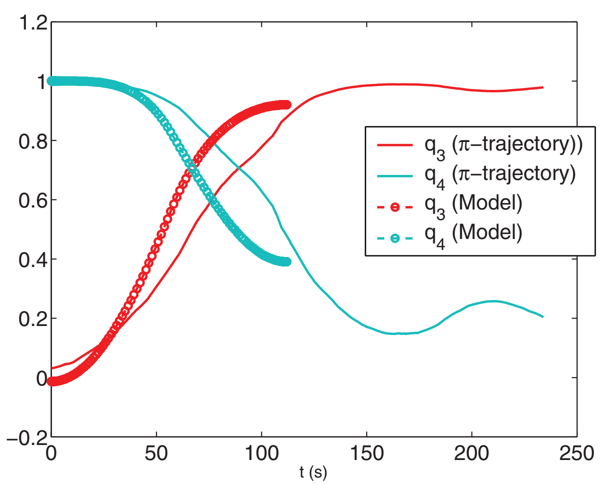


**Fig. 14 Failed closed-loop performance of NPSAT1 to a sample-and-hold feedback control.**

### E. $\pi$-Trajectory Generation: A Brief Discussion on Experimental Results

Although PS methods may be used to generate Carathéodory–$\pi$ trajectories, they may also be used to implement a sample-and-hold feedback controller. It is worth comparing the practical performances of the clock-based and sample-and-hold feedback controllers on the NPSAT1 testbed when both feedback controllers are generated using the same PS method. The experimental result of the system response to a sample-and-hold feedback controller is shown in Fig. 14. It is quite clear that this controller, despite being PS-generated, has failed. Although not shown here for the purpose of brevity, it can be numerically demonstrated that a sampled data feedback controller would perform successfully if the computation time were to be substantially faster. Thus, the requirements on real-time computation are indeed high when the feedback is in a sample-and-hold form, but ironically, the computational burden on our proposed clock-based feedback implementation is indeed low. This amplifies the arguments of Sec. V, particularly, Remark 2 of Lemma 1. In emphasizing the connections between theory and experiment, note that the closed-loop trajectory shown in Fig. 14 is a $\pi$ trajectory, whereas results reported in Sec. VII.E are Carathéodory–$\pi$ trajectories. As an illustrative point of comparison, we conclude this section with Fig. 15, wherein a sample-and-hold implementation for $t \mapsto u_1$ is plotted alongside a clock-based feedback implementation. Comparing the plots, the stark differences in the control trajectories are readily apparent.

## VIII. Additional Examples

It is quite clear from Sec. VII that sophisticated feedback control laws are not only possible, but that they are practically implementable, at least through the use of PS methods. Although the experimental results reported in Sec. VII are the first of their kind, there is substantial evidence to indicate that these ideas are readily portable to virtually any nonlinear system that satisfies the mild conditions indicated in Sec. I. For example, in [48], optimal solutions to a robotics problem was obtained at approximately 30-Hz frequency. Feedback solutions (in the sense of Carathéodory–$\pi$) to a number of problems that include the classically difficult problem of swing-up of an inverted pendulum are presented in [6]. Reentry guidance for an X-33 class launch vehicle based on the same principles described in this paper is discussed in [5]. Solutions that satisfy the real-time computational conditions set forth in Sec. V have been reported for such diverse problems as spacecraft formation design and control [59], low-thrust trajectory optimization [60], singularity-free steering of control moment gyros [61], and loitering of unmanned air vehicles , to name just a few. Almost all of the computations reported in these papers and elsewhere were performed with unoptimized versions of the DIDO software package [50] running in a MATLAB environment under a Microsoft Windows operating systems. Despite such runtime overhead, PS controls have been computed at a sufficiently fast sampling rate for all these systems. These recent results, together with the experimental demonstrations reported in this paper, show that optimal feedback controls for complex nonlinear systems are demonstrable under adverse conditions and available computer technology. Under such a framework, the grand challenges in control theory are sharply focused at its very foundations as theory and computation continue to merge at the level of first principles itself. We are therefore able to ask more fundamental questions about the meaning of a solution, the purpose of feedback, the proper framework for optimal control, the role of advance computing methods, the interplay between these topics, and much more. An excellent overview of such fundamental

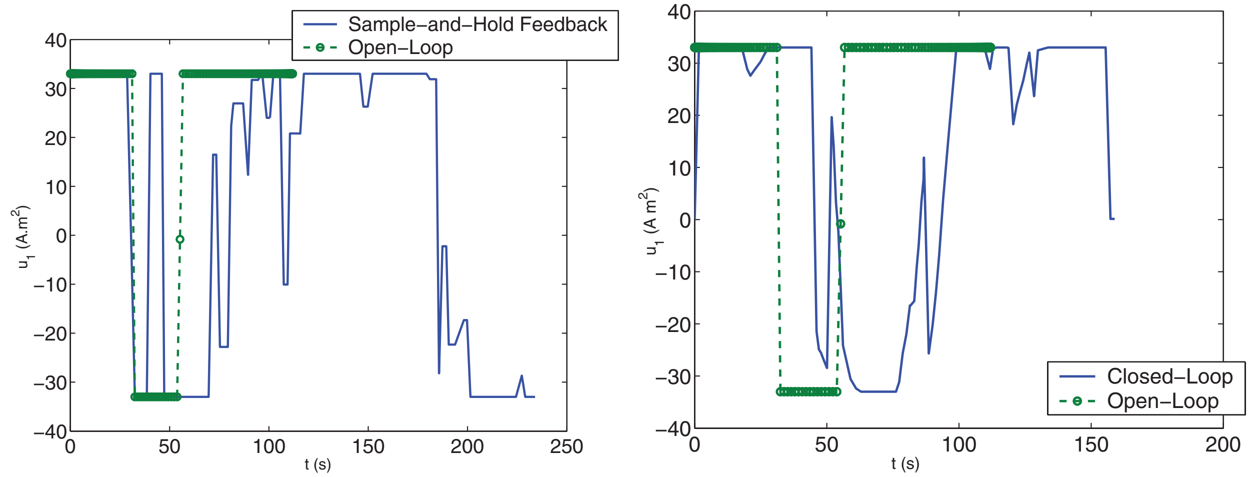


**Fig. 15 Failed sample-and-hold (left) and successful clock-based (right) feedback controls generated using PS methods.**

theoretical problems is described by Clarke [16] (see also [2,28,52–54]).

## IX. Conclusions

Addressing practical problems related to safety, fault tolerance, etc., by way of merely feasible feedback controllers creates unintended new problems. Addressing these new problems makes the design of the closed-loop system more complex than necessary. Optimal feedback controllers are not only superior in performance to feasible feedback controllers, they are frequently simpler to design; however, the standard route of finding them by way of the Hamilton–Jacobi–Bellman (HJB) framework poses significant theoretical and practical problems beyond Bellman's famous "curse of dimensionality." For many years now, these problems have relegated the use of the HJB theory to simple or academic-strength problems. The new approach to optimal feedback control based on the Carathéodory–$\pi$ trajectory concept is capable of solving industrial-strength problems. This concept was borne from the elementary observation that the more fundamental problem to feedback control is the generation of *closed-loop,* and not *closed-form*, solutions to $u = k(t, x)$. An analysis of this observation reveals that the Lipschitz constant of the dynamics provides a means to measure a fundamental sampling frequency for constructing closed-loop solutions. Despite its mathematical sophistication, the Carathéodory–$\pi$ solution concept is quite simple to implement, particularly through the use of pseudospectral (PS) methods. When these ideas are combined with optimal control theory, feedback representations are viewed as designer functions, $k: \mathbb{R} \times \mathbb{R}^{N_x} \to \mathbb{R}^{N_u}$, with optimal control theory providing the design framework. Given that PS methods can demonstrably generate control trajectories in fractions of a second to a few seconds, it is easy to conclude that a new approach to effective optimal feedback control is indeed possible for a rich variety of nonlinear dynamic systems. The successful implementation of these concepts for the ground test of the NPSAT1 attitude control, and the recent flight demonstration onboard the International Space Station indicates their practicality. These breakthroughs open up a rich variety of basic questions that form the new fundamentals for feedback control. These fundamentals lie at the intersection of optimal control theory, nonsmooth analysis, optimization, functional analysis, and computational principles.

## Appendix: Mathematical Foundations

### I. Carathéodory Solution

An initial value (Cauchy) problem,

$$\dot{x} = g(x, t), \qquad x(t_0) = x^0$$

where $g(x, t)$ is a time-varying vector field, $g: \mathbb{R}^{N_x} \times \mathbb{R} \to \mathbb{R}^{N_x}$ is said to have a Carathéodory solution [21,63] if there exists an absolutely continuous function, and $x^*(\cdot): R \supset [t_0, t_f] \to \mathbb{R}^{N_x}$, such that

$$\dot{x}^*(t) = g(x^*(t), t) \quad \text{for } a.a.\, t \in [t_0, t_f] \quad \text{and} \quad x^*(t_0) = x^0$$

### II. $L^1$-Carathéodory Conditions

Let $g(x, t)$ be a time-varying vector field, $g: \mathbb{R}^{N_x} \times \mathbb{R} \to \mathbb{R}^{N_x}$, defined over a tube

$$\mathbb{T}(\xi, \epsilon) := \{(x, t): t_0 \le t \le t_f, \|x - \xi(t)\| \le \epsilon\}$$

around an absolutely continuous function $\xi: [t_0, t_f] \to \mathbb{R}^{N_x}$. If $g(x, t)$ is

1) *measurable* in $t$ for each $x$,
2) *continuous* in $x$ for almost all $t$, and
3) $\|g(x, t)\| \le \varrho(t)$ for each $x$ and almost all $t$ and some $\varrho \in L^1([t_0, t_f], \mathbb{R}^{N_x})$,

then $g$ is said to satisfy $L^1$-Carathéodory conditions [63].

### III. Carathéodory Existence Theorem

If $g$ is $L^1$-Carathéodory, then there exists a Carathéodory solution [21,63] for the initial value problem: $\dot{x} = g(x, t)$, $x(t_0) = x^0$ if $(x_0, t_0) \in \mathbb{T}(\xi, \epsilon)$.

### IV. $C^1$-Carathéodory Conditions

Let $g$ be $L^1$-Carathéodory. If

1) $g(x, t)$ is $C^1$ in $x$ for almost all $t$,
2) $\partial_x g(x, t)$ is measurable in $t$ for each $x$, and
3) $\|g(x, t)\| + \|\partial_x g(x, t)\| \le \varrho(t)$ for all $(x, t) \in \mathbb{T}(\xi, \epsilon)$

then $g$ is said to satisfy $C^1$-Carathéodory conditions [33].

### V. Lambert $W$ function

The multivalued function $\mathbb{R} \ni x \twoheadrightarrow W(x)$, given implicity by

$$x = W(x)e^{W(x)} \tag{A1}$$

is called the Lambert $W$ function [64]. A detailed description of this function, along with its historical origins and many applications, is described in [64]. For $x \ge 0$, $W(x)$ is single-valued.

### VI. Gronwall's Lemma

Let $[t_0, t_f] \mapsto y(t) \in \mathbb{R}_+$ be an integrable function that satisfies Gronwall's inequality [65]:

$$y(t) \le a(t) + \int_{t_0}^{t} b(s) y(s) ds$$

where $a$ and $b$ are continuous, nonnegative, bounded functions, with $t \mapsto a(t)$ nondecreasing over the interval $[t_0, t_f]$; then

$$y(t) \le a(t)e^{B(t)}$$

where

$$B(t) := \int_{t_0}^{t} b(s)ds$$

### VII. Sobolev Spaces

The Sobolev space [66] $W^{m,p}([t_0, t_f], \mathbb{R}^n)$, or simply $W^{m,p}$ for short, is the space of all functions $\xi$: $[t_0, t_f] \to \mathbb{R}^n$, for which the $j$th distributional derivative $\xi^{(j)}$ lies in $L^p([t_0, t_f], \mathbb{R}^n)$ for all $0 \le j \le m$ with the norm

$$\|\xi\|_{W^{m,p}} = \sum_{j=0}^{m} \|\xi^{(j)}\|_{L^p}$$

Thus, $W^{1,\infty}$ denotes the space of functions with bounded first derivatives.

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